

مهارات التفكير العليا

التكامل بالكسور الجزئية

تبرير: أحل السؤالين الآتيين تباعاً:

(33) أجد: $\int dx \sqrt{1+e^x}$ بطريقتين مختلفتين، إحداهما الكسور الجزئية، مبرراً أجابتي.

الحل الأول بضرب كل من البسط والمقام بـ e^{-x}

$$\int (e^{-x}+1)+C e^x dx = \int e^{-x} e^{-x} + 1 dx = -\int e^{-2x} + 1 dx = -\ln|1+e^{-x}| + C$$

الحل الثاني بالتعويض:

$$u = e^x \Rightarrow du = e^x dx = u dx \Rightarrow dx = \frac{du}{u} \int \sqrt{1+e^x} dx = \int \sqrt{1+u} \times \frac{du}{u} = \int \frac{\sqrt{1+u}}{u} du$$

$$\frac{\sqrt{1+u}}{u} = \frac{A}{u} + \frac{B}{\sqrt{1+u}} \Rightarrow 1 = A\sqrt{1+u} + Bu \Rightarrow A = -1, B = 1$$

$$\int \frac{\sqrt{1+u}}{u} du = \int \left(\frac{-1}{u} + \frac{1}{\sqrt{1+u}} \right) du = -\ln|u| + \ln|1+u| + C$$

$$= -\ln(e^x) + \ln(1+e^x) + C = \ln\left(\frac{1+e^x}{e^x}\right) + C = \ln\left(1 + \frac{1}{e^x}\right) + C$$

(34) أجد: $\int \frac{1}{1+e^x} dx$

$$\int \frac{1}{1+e^x} dx = \int \frac{1}{1+u} \frac{du}{u} = \int \frac{1}{u(1+u)} du = \int \left(\frac{A}{u} + \frac{B}{1+u} \right) du$$

$$1 = A(1+u) + Bu \Rightarrow A = 1, B = -1$$

$$\int \frac{1}{u(1+u)} du = \int \left(\frac{1}{u} - \frac{1}{1+u} \right) du = \ln|u| - \ln|1+u| + C = \ln\left|\frac{u}{1+u}\right| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

(35) تبرير: أثبت أن: $\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \ln|3x-2| + \frac{1}{x-1} + C$

$$\frac{5x^2 - 8x + 12}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 5x^2 - 8x + 12 = A(x-1) + B$$

$$5x^2 - 8x + 12 = Ax - A + B \Rightarrow A = 5, B = 17$$

$$\int \frac{5x^2 - 8x + 12}{(x-1)^2} dx = \int \left(\frac{5}{x-1} + \frac{17}{(x-1)^2} \right) dx = 5 \ln|x-1| - \frac{17}{x-1} + C$$

(36) تبرير: أثبت أن: $\int \frac{3x^2 - 4}{(x^2+1)^2} dx = \frac{3}{2} \arctan(x) - \frac{2}{x^2+1} + C$

$$u=x \Rightarrow u^2=x \Rightarrow dx=2u du \Rightarrow x=9 \Rightarrow u=3 \Rightarrow x=16 \Rightarrow u=4 \int \frac{9-16}{2x^2} dx = \int \frac{34}{2u^2} du = \int \frac{17}{u^2} du = \int 17u^{-2} du = \frac{17}{-1} u^{-1} = -17u^{-1} = -\frac{17}{u} = -\frac{17}{\sqrt{x}}$$

(37) تبرير: أثبت أن: $\int \frac{5x^2+9x+4}{x^2+2x+3} dx = 2 + 12 \ln|x+3| - 5 \ln|x+1| + C$

$$\frac{5x^2+9x+4}{x^2+2x+3} = \frac{5x^2+9x+4}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3} \Rightarrow 5x^2+9x+4 = A(2x+3) + B(x+1)$$

$$5x^2+9x+4 = 2Ax+3A+Bx+B \Rightarrow 5x^2+9x+4 = (2A+B)x + (3A+B)$$

$$\begin{cases} 2A+B=9 \\ 3A+B=4 \end{cases} \Rightarrow \begin{matrix} 2A+B=9 \\ -A=5 \end{matrix} \Rightarrow \begin{matrix} A=-5 \\ B=19 \end{matrix}$$

$$\int \frac{5x^2+9x+4}{x^2+2x+3} dx = \int \left(\frac{-5}{x+1} + \frac{19}{2x+3} \right) dx = -5 \ln|x+1| + \frac{19}{2} \ln|2x+3| + C$$

تحذ: أجد كلاً من التكاملات الآتية:

(38) $\int \frac{1+x}{x^2} dx$

$$\frac{1+x}{x^2} = \frac{1}{x^2} + \frac{x}{x^2} = x^{-2} + x^{-1}$$

$$\int \frac{1+x}{x^2} dx = \int x^{-2} dx + \int x^{-1} dx = \frac{x^{-1}}{-1} + \ln|x| + C = -\frac{1}{x} + \ln|x| + C$$

(39) $\int \frac{16x^4-1}{x^2} dx$

$$\frac{16x^4-1}{x^2} = \frac{(4x^2+1)(2x-1)(2x+1)}{x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{4x^2+1} + \frac{D}{2x-1} + \frac{E}{2x+1}$$

$$16x^4-1 = A(4x^2+1)(2x-1)(2x+1) + B(2x-1)(2x+1) + Cx^2(2x-1)(2x+1) + Dx^2(4x^2+1) + Ex^2(2x-1)(2x+1)$$

$$16x^4-1 = A(4x^2+1)(4x^2-1) + B(4x^2-1) + Cx^2(4x^2-1) + Dx^2(4x^2+1) + Ex^2(4x^2-1)$$

$$16x^4-1 = (4A+4C+4E)x^4 + (4A+B) x^2 - (A+B) = 16x^4 - 1$$

$$\begin{cases} 4A+4C+4E=16 \\ 4A+B=0 \\ -A-B=-1 \end{cases} \Rightarrow \begin{matrix} A+C+E=4 \\ 4A+B=0 \\ -A-B=-1 \end{matrix} \Rightarrow \begin{matrix} A=1 \\ B=-5 \\ C=3 \end{matrix}$$

$$\int \frac{16x^4-1}{x^2} dx = \int \left(\frac{1}{x} - \frac{5}{x^2} + \frac{3}{4x^2+1} \right) dx = \ln|x| + \frac{5}{x} + \frac{3}{2} \arctan\left(\frac{x}{2}\right) + C$$

$$\int (2x+1)|2x-1| + 116 \ln(4x^2+1) + 116 \ln|2x-1| + 182x+1) dx = -116 \ln|4x^2-14x^2+1| + CC = 116 \ln$$

$$\int (1x-x^3) dx \quad (40)$$

$$u = x^6 \Rightarrow du = 6x^5 dx \Rightarrow dx = \frac{du}{6x^5} = \frac{du}{6u^{5/6}} = \frac{1}{6} u^{-5/6} du$$

$$\int (1x-x^3) dx = \int (u^{1/6} - u^{3/6}) \cdot \frac{1}{6} u^{-5/6} du = \frac{1}{6} \int (u^{1/6-5/6} - u^{3/6-5/6}) du = \frac{1}{6} \int (u^{-2/3} - u^{-2/6}) du$$

$$= \frac{1}{6} \int (u^{-2/3} - u^{-1/3}) du = \frac{1}{6} (3u^{1/3} - 3u^{2/3}) + C = \frac{1}{2} (u^{1/3} - u^{2/3}) + C$$

$$= \frac{1}{2} (x^2 - x^3) + C$$